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Hyperspin Manifolds

David Finkelstein,[1] Shlomit Ritz Finkelstein,[1] and Christian Holm[1]

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Riemannian manifolds are but one of three ways to extrapolate from four-dimensional Minkowskian manifolds to spaces of higher dimension, and not the most plausible. If we take seriously a certain construction of time space from spinors, and replace the underlying binary spinors by N-ary hyperspinors with new "internal" components besides the usual two "external" ones, this leads to a second line, the hyperspin manifolds $\mathfrak{S}_N$ and their tangent spaces $d\mathfrak{S}_N$, different in structure and symmetry group from the Riemannian line, except that the binary spaces $d\mathfrak{S}_2$ (Minkowski time space) and $\mathfrak{S}_2$ (Minkowskian manifold) lie on both. $d\mathfrak{S}_N$ and $\mathfrak{S}_N$ have dimension $n = N^2$. In hyperspin manifolds the energies of modes of motion multiply instead of adding their squares, and the N-ary chronometric form is not quadratic, but N-ic, with determinantal normal form. For the nine-dimensional ternary hyperspin manifold, we construct the trino, trine-Gordon, and trirac equations and their mass spectra in flat time space. It is possible that our four-dimensional time space sits in a hyperspin manifold rather than in a Kaluza-Klein Riemannian manifold. If so, then gauge quanta with spin-3 exist.

1. INTRODUCTION

1. Premises. Since the work of Kaluza it has come to be widely surmised that ordinary time space is imbedded in a higher dimensional space popularly called hyperspace, and that all the interactions in nature are manifestations of hyperspace curvature, the curvature being that of ordinary time space for gravity and of other dimensions for other forces. These other dimensions probably have microscopic quantum extent rather than macroscopic extent, and are therefore generally called *internal*; then the ordinary ones are *external.* Just as there are at least implicit and heuristic quantization rules that assist us in building quantum theories from classical ones, there

[1]Georgia Institute of Technology, Atlanta, Georgia 30332.

are similar *imbedding strategies* for building internal space out of external, such as the following:

2. Riemannian strategy. Add new internal components to the ordinary external ones of time space vectors (and differentials), preserving the general Riemannian structure of differential geometry.

The theories and concepts resulting from such an imbedding we also call Riemannian, although, of course, Riemann thought in definite Euclidean forms, while the Minkowski quadratic form needed for physics is indefinite. The Riemannian strategy, with many variations, is the one that has long been followed to higher dimensional geometries, such as the Kaluza-Klein five-dimensional theory of electromagnetism and De Witt's (1964) $(4+n)$-dimensional theories of higher gauge fields. The Riemannian strategy, however, conflicts with a principle put forward long ago by v. Weizsäcker (1955) and Penrose (1970), among others:

3. Spinor principle. The fundamental entities of nature are not time space vectors or differentials, but are described by spinors.

Spinors stand out because they support the fundamental representation of the Lorentz group, because of their quantum significance, and because their metrical geometry, being purely affine, based purely on a relative volume element, does not reduce the affine symmetry of the linear space of spinors. The only spinor structure that is necessary for the construction of time space is their Grassmann algebra. When vectors instead of spinors are taken as basic, they must be given "from outside" an indefinite quadratic form, the proper time; when vectors are composed of spinors, however, their quadratic form is determined by their composition.

From this conflict of principles emerges the following:

4. Spinor strategy. Add new internal components to the two external one of time space spinors, preserving the general algebraic structure of the differential geometry.

If spin is prior to space, then hyperspin is prior to hyperspace. Binary spinors generate the Minkowskian time space manifold, which we now call binary. We compute the time space generated by a spinor of N components, a hyperspinor. The result is a new N-ary spin, or hyperspin, manifold, in which the binary is naturally imbedded.

The spin manifolds now form an infinite sequence of higher-dimensional geometries $\mathfrak{S}_N \equiv \mathfrak{S}^n = \mathfrak{S}^{N^2}$ (exponents give the dimension n of the space; subscripts give the dimension N of the fundamental spinors giving rise to the space) quite different in structure from the Riemannian manifolds $\mathfrak{R}^n$, except for $N=2$, where T_2 is a Minkowskian $\mathfrak{R}^4$.

Riemannian geometries have real orthogonal groups $O(n_+, n_-, \mathbb{R})$ for their local invariance groups, with various signatures, belonging to Cartan's B_m and D_m, but the hyperspin manifold T^n has the much smaller group $GL(\sqrt{n}, \mathbb{C})$ instead, belonging to Cartan's A_{N-1}, and not to be confused with the group $GL(n, \mathbb{R})$ of manifolds.

Riemannian manifolds have a quadratic chronometric form. (We reserve the word "metric" for the quantum Hilbert space.) The N-ary spin manifold has an N-ic chronometric form.

5. Paraspin and hyperspin. To reduce the possibility of misunderstanding, we distinguish between hyperspin and paraspin. We call a "paraspin" any spinlike degree of freedom, like isospin or color, that is not a spin at all, in that it does not act like the Lorentz group on time space elements. Hyperspin, however, extends and includes ordinary spin. One difference between paraspin and hyperspin is that between multiplying and adding a new space to the binary spinor space, between $N_1 N_2$ and $N_1 + N_2$ spinorlike components.

6. Ternary spinors and *N*-ary. Our first extension turns two-component spinors into three-component ones, not by multiplying degrees of freedom, since 3 does not contain 2 as a factor, nor through any Riemannian geometry, whose spinors have a dimension that is always a power of 2, but simply by adding one hyperspinor component. In the following we treat this extension most fully. It takes us from a four-dimensional quadratic chronometric to a nine-dimensional cubic one. The generalization from 3 to N is clear. Anything labeled binary in this paper is thus ordinary and part of the old world, the correspondence limit; anything labeled ternary represents a whole new world.

This work is a by-product of a program in quantum topology that suggests that the case of large N is of physical importance, too. The first step in non-Euclidean geometry was to replace zero curvature by a different constant curvature; only later was it considered that the curvature might be a physical variable. Some presently propose a different constant dimension to take the place of 4; we consider dimension, too, as a physical variable, and are concerned with geometries of arbitrarily high dimension.

7. Course of action. Carrying out the plan of paragraph 4, we first choose a definite binary path from the theory of 2-spinors (stage i) through the theory of flat time-space geometry (stage ii) and curved (stage iii) to Einstein's theory of gravity (stage iv), which is set in the manifold that is both a spin manifold and Riemannian (Section 2, especially paragraphs 12–16). Then in Section 3 we replace all the 2-spinors along this path by

3-spinors (or N-spinors). The new path (paragraphs 18–25) then leads us to a hyperspin manifold, imbedding the binary spin manifold, and quite off the line of Riemannian geometries. In paragraph 26 we work out the simplest invariant wave equations of the ternary spin manifold. Section 4 summarizes the results and possible alternate paths.

8. Correspondence limit. We think of the components of our fundamental spinors as quantum amplitudes. The work is guided by a still tentative correspondence principle that an effectively four-dimensional time space physics results when the extent ε of the internal space goes to zero relative to the other independent lengths of the experiment, for it then takes too much energy $[O(\hbar c/\varepsilon)]$ to excite components of momentum along the internal dimensions. Ordinary physical quantities therefore should not depend appreciably on the internal variables. As $\varepsilon \to 0$, internal derivatives $\partial/\partial\psi^3, \ldots$ should approach 0 relative to external ones $\partial/\partial\psi^1$ and $\partial/\partial\psi^2$.

In practice, since we have only indirect evidence about what goes on in internal space, in each case we seek anew the conditions that must be imposed on the internal behavior to reproduce the binary external behavior. The existence of such a correspondence limit is assured by the spinor principle (paragraph 3) and the imbedding of 2-spinors in hyperspinors.

9. Spinors and hyperspinors. In general we call a *spinor* space a complex linear N-space Σ^N transforming under the group $SL(N, \mathbb{C})$, consistent with the identification of two-component spinors of $\Sigma^2 \subset \Sigma^N$ as ordinary Weyl 2-spinors, and of $SL(2, \mathbb{C}) \subset SL(N, \mathbb{C})$ as ordinary time space rotations and Lorentz transformations. That is, if $N > 2$, the ordinary Weyl 2-spinors must be identified with a subspace $\Sigma^2 \subset \Sigma^N$. Those N-ary spinors with $N > 2$ are hyperspinors.

The spinors introduced here support the unimodular groups $A_1, A_2, \ldots,$ and may therefore be called *unimodular spinors* to distinguish them from the orthogonal spinors of Cartan, which support the orthogonal groups B_m and D_m. It is somewhat bizarre to take orthogonal hyperspinors as more fundamental than hyperspace, for they have exponentially more numerous components than the vectors they define: $N = O(2^{n/2})$. Unimodular spinors have much fewer components than their vectors: $N = \sqrt{n} = o(n)$. Symplectic spinors supporting the symplectic groups C_m appear briefly in paragraph 31.

10. Polyspinor algebra. We designate the Grassmann algebra over any linear space L by $\{\mathbf{L}\}$. If ψ is an element of L, we designate the corresponding first-grade element of $\{\mathbf{L}\}$ by $\{\boldsymbol{\psi}\}$. An index of $\{\boldsymbol{\psi}\}$ may be taken to be a *set* of indices of ψ arranged in numerical order. We designate a variable taking on such values by $\{\mathbf{A}\}$, where A is an index variable of ψ. For example, if $A = 1, 2$ then $\{\mathbf{A}\} = \{\ \}, \{\mathbf{1}\}, \{\mathbf{2}\}, \{\mathbf{12}\}$. We may drop the condition that the values be in numerical order if we impose antisymmetry instead.

Evidently $\{\boldsymbol{\psi}\}^2 = 0$ regardless of the value of ψ^2. Vectors in boldface braces anticommute. The basis element $\{\mathbf{e}\}_{\{\}}$ of $\{\mathbf{L}\}$ is generally called the *vacuum* (of that basis); it is the unit of the Grassmann algebra, and is generally designated by 1. We therefore call the basis element of $\{\mathbf{L}\}$ that is the product of the braces of all the basis elements of L the *plenum* (of that basis), designating it by $U_{\{\mathbf{L}\}}$ or simply U:

$$U = \{e_1\} \cdots \{e_n\}, \qquad U^2 = 0$$

When 1 represents the empty state, U represents the full state. In Grassmann's interpretation, 1 represents the empty set and U represents the unit cell of the basis. In quantum set theory, 1 represents the empty set and U the universal set.

Indices. Dual spaces and vectors of a dual basis carry the superscript D and prefix co-. Spinor indices are capital Greek letters. The indices of conjugate spinors are dotted: $\dot{A} = \dot{1}, \dot{2}$. Both A and $\dot{A}$ are independently variable index symbols unless otherwise stated. Time space indices are lower case Latin letters. Lower case Greek indices stand for a pair of corresponding spinor indices, one dotted, as follows:

$$\varphi^a := \varphi^{A\dot{A}}, \quad \varphi^b := \varphi^{B\dot{B}}, \ldots; \qquad \varphi_a := \varphi_{\dot{A}A}, \quad \varphi_b := \varphi_{\dot{B}B}, \ldots$$

Polyspinors. We now have the following tensorial operations for assembling new linear spaces from old: the direct sum (+) and product ($\otimes$); conjugate (superscript C) and dual (superscript D); and $\{\,\}$. When we multiply algebras with $\otimes$, we postulate that the factors commute unless both are Grassmann algebras, when they anticommute. We call the iterates of these tensorial operations *polytensorial.* The linear spaces they generate from a given linear space L are the *polytensors* of L. From spinors they generate *polyspinors*; from vectors, *polyvectors*; from monadics, *polyadics*, including the usual dyadics as a highly special case. Polytensors are tensors with an additional hierarchic structure.

Some polyvectors over L are invariant under a sign reversal in L, and called *even*, and some change sign and are called *odd.* We call this parity the linear (group) parity. Under a phase change $\lambda \to e^{i\theta}\lambda$ of all $\lambda \in L$, a polyvector p over L that transforms into $e^{iQ\theta}p$ is said to have linear (group) charge Q over L. The linear parity is the parity (evenness or oddness) of the linear charge.

Polyforms. When a nonsingular bilinear form $b = b(\varphi, \psi)$ is given both on $L \otimes L \to \mathbb{C}$ and on the dual space $L^{\mathrm{D}} \otimes L^{\mathrm{D}} \to \mathbb{C}$, there exists a useful extension to all polytensors over L respecting the tensorial operations. If $T(ABC\ldots)$ and $U(ABC\ldots)$ are two polytensors with the same series of indices (each of which may be high or low, dotted or undotted), we define

$$b(T, U) = b(AA')b(BB')b(CC')\ldots T(ABC\ldots)U(ABC\ldots)$$

where each $b(\ldots)$ represents a bilinear form of the kind indicated by its indices; that is, the dual form b^{D} if the indices are raised, conjugated if they are dotted. If T and U have different index structures, we set

$$b(T, U) = 0$$

We call $b(T, U)$ the polyform induced by b on the polytensor T.

N-linear forms induce similar polyforms. Suppose $b(\psi_1, \ldots, \psi_N)$ is such a form on $L^{\otimes N} \to \mathbb{C}$ and b^{D} on $(L^{\mathrm{D}})^{\oplus N} \to \mathbb{C}$. Then we define $b(T_1, \ldots, T_N)$ for any tensors $T_1, \ldots, T_N$ by

$$b(T_1, \ldots, T_N) = b(A_1 \ldots A_N)b(B_1 \ldots B_N) \ldots b(C_1 \ldots C_N)T_1(ABC\ldots)\ldots T_N(ABC\ldots)$$

where each $b(A_1 \ldots A_{\mathrm{N}})$ represents an N-linear form of the kind indicated by its indices; that is, b is replaced by b^{D} if the indices are raised, and conjugated if they are dotted.

2. BINARY SPIN MANIFOLDS

11. One path from spin to gravity. The following stages i–iv are each familiar (see p. 212 of Penrose and Rindler (1984)), but we set them down in order to work on them. They happen to bypass torsion, which may well exist in nature; we take this path anyway for brevity.

12. Stage i. Spinors. Start from the space Σ^2 (the boldface for 2 is explained in paragraph 17) of *binary spinors* $\psi = (\psi^A) = e_A\psi^A$, $A = 1$, $\mathbf{2}$, isomorphic to the linear space $\mathbb{C}^2$ of **pairs** of complex numbers, having fundamental symmetry group $SL = SL(\mathbf{2}, \mathbb{C})$, typical basis elements e_A, and special basis elements

$$1_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \uparrow \quad \text{(read "up")}$$

$$1_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \downarrow \quad \text{(read "down")}$$

Form the complex Grassmann algebra $\{\Sigma^2\}$ over Σ^2. We designate the vacuum and plenum of $\{\Sigma^2\}$ by 1 and U_2, respectively, with $(U_2)^2 = 0$.

Designate the *grade* of $\{\Sigma^2\}$ by Γ; Γ is the linear operator on $\{\Sigma^2\}$ with eigenvectors 1, $\{e_1\}$, $\{e_2\}$, and U_2, and corresponding eigenvalues 0, 1, 1, and 2. In a quantum interpretation of spinors as creators, Γ is the number of things created, usually called the occupation number; in Grassmann's interpretation, Γ is the dimension of the element of extension.

The Grassmann product $\{\boldsymbol{\varphi}\}\{\boldsymbol{\psi}\}$ is a complex multiple of the plenum U for any spinors φ, ψ. It may therefore be used to define a complex antisymmetric bilinear form $\delta(\varphi, \psi) =: \delta$ on $\Sigma^2 \otimes \Sigma^2 \to \mathbb{C}$, through

$$\{\boldsymbol{\varphi}\}\{\boldsymbol{\psi}\} = \boldsymbol{\delta}(\boldsymbol{\varphi}, \boldsymbol{\psi}) U$$

This form is just the determinant of the two spinors involved. δ is a scalar under the unimodular group.

The polyform construction of paragraph 10 extends δ to a polyform $\delta(p, q)$ defined for polyspinors p and q of any polyspinor. We will call δ and its polyforms the *binary forms.* We define a *binary norm* $\|p\|$ for any polyspinor p by

$$\|p\| = \delta(p, p)$$

This norm is a generalization to multidimensional hypercubical matrices of the determinant of a two-dimensional square matrix. It vanishes by antisymmetry for odd polyspinors.

13. Stage ii. The tangent time space. Form from Σ^2 the **four**-dimensional complex linear space $\mathbb{A}_2 := \Sigma^2 \otimes \Sigma^{2C}$, the tensor product of the spinor space and its conjugate. We call this $\mathbb{A}_2$. The complex tangent spaces to time space, the fibers of the time space tangent-vector bundle, will be isomorphs of this space. Call its elements *ambispinors.* Writing them as

$$\mathbf{t} = (t^{A\dot{A}}) =: (t^a)$$

Under group elements $\Lambda \in SL(2, \mathbb{C})$, a spinor ψ and ambispinor $\mathbf{t}$ transform according to

$$\psi' = \Lambda \psi, \qquad \mathbf{t}' = \Lambda \mathbf{t} \Lambda^{\mathrm{H}}$$

$\mathbb{A}_2$ possesses a natural *adjoint* operation ${}^{\mathrm{A}}: \mathbf{t} \to \mathbf{t}^{\mathrm{A}}$ that is defined to be the real-linear operator with $(\varphi \otimes \psi^{\mathrm{C}})^{\mathrm{A}} = \psi \otimes \varphi^{\mathrm{C}}$. *Real time space tangent vectors* are identified with *self-adjoint* elements of $\mathbb{A}_2$:

$$\mathbf{t}^{\mathrm{A}} = \mathbf{t}$$

Cones. Identify *future timelike* tangent vectors with positive-definite self-adjoint $\mathbf{t} = \mathbf{t}^{\mathrm{A}} > 0$, forming the *future cone* $\mathbb{S}^{++}$.

Except for boundary cases, every tangent vector $\mathbf{t}$ falls into one of the three classes $\mathbb{S}^{++}$, $\mathbb{S}^{+-}$, $\mathbb{S}^{--}$; the superscripts give the signs of the eigenvalues of the normal form of $\mathbf{t}$. These are future timelike, spacelike, and past timelike cones. The boundary cases are the future null cone $\mathbb{S}^{+0}$, the past null cone $\mathbb{S}^{0-}$, and the origin $\mathbb{S}^0$; the superscripts give the **two** diagonal elements (+1, 0, or −1) of the Sylvester normal form of **t** in nonincreasing order.

Derivative. We use the familiar invariant differential operator

$$\boldsymbol{\partial} = (\partial_{\dot{B}B}) := (\partial/\partial t^{B\dot{B}})$$

acting on scalar functions $\varphi(\mathbf{t})$ on $\mathbb{S}_2$.

14. The chronometric form. Define the *chronometric* (*quadratic*) *form* on $\mathbb{S}_2$ as the N-ary norm δ extended to $\mathbb{S}_2$, applying the polyform construction of paragraph 10 to the bilinear form $b = \delta$, the linear space $L = \Sigma^2$, and $\mathbb{S}_2$:

$$\|\mathbf{t}\|_2 = 2\det(\mathbf{t}) = \delta_{ab}t^a t^b, \qquad \delta_{ab} := \varepsilon_{AB}\varepsilon_{\dot{A}\dot{B}}$$

The space $\mathbb{S}_2$ is provided with the **quadratic** form $\|\ldots\|_2$. The space $\mathbb{S}_2$ has one timelike dimension. This is seen when the norm is diagonalized by means of the coordinates t, x, y, z of Cartan,

$$\mathbf{t} = \begin{bmatrix} t+z & x-iy \\ x+iy & t-z \end{bmatrix} / 2^{1/2}$$

$$\|\mathbf{t}\|_2 = 2\det(\mathbf{t}) = t^2 - x^2 - y^2 - z^2$$

15. Stage iii. The spin manifold. Assume that world is a *binary spin manifold* or $\mathfrak{S}_2$; that is, is locally isomorphic to the binary space $\mathbb{S}_2$ of stage ii. We mean by this that at each point there exists a preferred coordinate system, called normal, where to lowest differential order the concepts and theories of $\mathbb{S}_2$ are valid. (This strict form of equivalence principle eliminates torsion.) The spin manifold therefore has the following familiar furniture at each point p:

1. A local spinor space $\Sigma^2(p)$.
2. A local tangent space $d\mathfrak{S}_2(p)$ isomorphic to $\mathbb{S}_2$, and an isomorphism σ, the *spin* map, inducing:
3. A real, symmetric **quadratic** form $g_{ab}(p)dt^a dt^b$, with coefficients $g_{ab} = \delta_{ab}$ in a normal coordinate system, inducing a local norm $\|\ldots\|_p$ for polyspinors.
4. A covariant derivative $\mathbf{D} = (D_a)$ generalizing the differential operator $\boldsymbol{\partial}$; a **2**-spinor *connection* $\Gamma^A{}_{Bc}(p)$ reducing to 0 at p in a normal coordinate system at p; and spinor curvature $R^A{}_{BCd}(p)$. The spinor connection and curvature induce the vector connection $\Gamma^a{}_{bc}$, curvature $R^a{}_{bcd}$, and Ricci tensor $R_{ab} := R^c{}_{acb}$ in a routine way.

16. Stage iv. Dynamics. Possible dynamical action density terms for **binary** gravity include the Hilbert action density $R\sqrt{g}$, with $R := R_{ab}g^{ab}$ and $g := -\det(g_{ab})$; the cosmological term $\sqrt{g}$; the action density

$$L_2 := \| R^A{}_{Bcd} \|_2 \sqrt{g}$$

suggested by gauge theory; and the determinantal action density of Einstein and Eddington

$$L_\Gamma := \det(R_{ab})^{1/2}$$

which is a function of the connection $\Gamma^a{}_{bc}$ alone, not involving the fundamental form g_{ab} at all. Actions for quanta moving in a manifold will be treated later.

3. TERNARY SPIN MANIFOLDS

17. Implementing the spinor strategy. Now we replace all **2**'s in these familiar postulates (having set them in boldface to make them easy to find) by 3's (and 4's by 9's), interpreting all resulting new time space coordinates as internal space dimensions.

The value of N (first 2, now 3) is the eigenvalue of the grade Γ on the eigenvector U, the plenum, of the Grassmann algebra over the space of spinors. The mathematical identity of N with a quantum occupation number for fermions leads us to conjecture that it indeed arises from such a quantity at a deeper, presently uncharted quantum level, and is a dynamical variable, not a constant of nature. We therefore carry out the extension from $N = 2$ to 3 in a way that can be extended to arbitrary N.

18. Stage i. Spinor space. First we replace Σ^2 by Σ^3, the space of ternary spinors, the three-dimensional complex linear space. The fundamental group of ternary spinors is $SL(3, \mathbb{C})$. The space Σ^3 supports the natural trilinear form δ defined by

$$\{\boldsymbol{\varphi}\}\{\boldsymbol{\psi}\}\{\boldsymbol{\xi}\} = \delta(\varphi, \psi, \xi)\, U = \varepsilon_{AB\Gamma}\varphi^A \psi^B \xi^\Gamma U$$

Here it multiplies the plenum $U = U_3$ of the Grassmann algebra $\{\boldsymbol{\Sigma}^3\}$. This form, with its analogously defined dual, extends to all polytensors over Σ^3 as in paragraph 10, and induces a trilinear norm $\|T\|_3$ for any such tensor:

$$\|T\|_3 := \delta(T, T, T)$$

When $SL(3, \mathbb{C})$ is reduced to $SL(2, \mathbb{C})$, a ternary spinor reduces to a superposition of a binary spinor and a binary scalar: $\Sigma^3 = \Sigma^2 + \Sigma^1$. The Σ^2 is identified with a subspace of Σ^3 called *external*. We write 3-spinor components as ψ^A $(A = 1, 2, 3)$ and 3-spinors themselves as one-column matrices or as $\psi = (\psi^A) = \mathrm{col}(\psi^1, \psi^2, \psi^3)$. The one-spinor basis consists of the special spinor symbols

$$\begin{bmatrix}1\\0\\0\end{bmatrix} = \uparrow \ \text{(read "up")}, \qquad \begin{bmatrix}0\\1\\0\end{bmatrix} = \downarrow \ \text{(read "down")}, \qquad \begin{bmatrix}0\\0\\1\end{bmatrix} = \rightarrow \ \text{(read "in")}$$

for two basic external spinors and one basic internal spinor. The ternary spinor frame is said to be *adapted* (to binary spin or time space) when the spinor component ψ^3 is 0 for external spinors. This is the *internal spin* component. The null coordinate $t^{33} = t^i$ is called the *internal* null coordinate. In an adapted spinor frame the *internal* null vector

$$o = e_i = (o^{A\dot{A}}) = (o^\alpha) := \rightarrow \rightarrow^c$$

has components

$$\begin{bmatrix}0 & 0 & 0\\0 & 0 & 0\\0 & 0 & 1\end{bmatrix}$$

19. Stage ii. The tangent time space. The ternary space $\mathbb{S}_3 \subset \Sigma^3 \otimes \Sigma^{3C}$ consists of 3×3 complex self-adjoint matrices

$$\mathbf{t} = (t^{A\dot{A}}) = \mathbf{t}^{\mathbf{A}}$$

and has nine complex dimensions. The differential operator

$$\boldsymbol{\partial} = (\partial_{\dot{B}B}) := (\partial/\partial t^{B\dot{B}})$$

acts on scalar functions $\varphi(\mathbf{t})$ on $\mathbb{S}_3$.

It follows that under transformations $\Lambda \in SL(3, \mathbb{C})$, 3-spinors ψ and 9-vectors $\mathbf{t}$ transform according to

$$\psi' = \Lambda\psi, \qquad \mathbf{t}' = \Lambda \mathbf{t} \Lambda^{\mathrm{H}}$$

$\mathbb{S}_3$ imbeds both the external binary space $\mathbb{S}_2$ with its external coordinates $x^0, \ldots, x^3 = t, x, y, z$ and an internal linear space I^5 of five spacelike dimensions, with internal coordinates $x^4, \ldots, x^8 = a, b, c, d, e$. We identify the matrices of $\mathbb{S}_2$ with the upper left 2×2 corner of the 3×3 matrices of $\mathbb{S}_3$ in an adapted frame, and I^5 with the border of this submatrix. The ternary tangent space $\mathbb{S}_3$ is provided with the fundamental form $\|\mathbf{t}\|_3 = \det(\mathbf{t})$. Its future timelike vectors $\mathbb{S}^{+++}$ are the positive-definite Hermitian $\mathbf{t}$, a convex invariant cone in $\mathbb{S}_3$.

Cones. Every ternary vector $\mathbf{t}$, except for boundary cases, falls into one of the four disjoint nine-dimensional classes $\mathbb{S}^{+++}, \mathbb{S}^{++-}, \mathbb{S}^{+--}, \mathbb{S}^{---}$ defined by the signs of the diagonal elements of the Sylvester normal form of $\mathbf{t}$. We call these classes *future timelike*, *future spacelike*, *past spacelike*, and *past* *timelike.* The boundary cases are represented by writing 0's in place of any of the $\pm$ signs in these symbols, indicating the presence of a diagonal element 0 in the normal form. Since the order of the superscripts is immaterial, we arrange them in nondecreasing order. There are therefore three disjoint lower dimensional null cones $\mathbb{S}^{++0}$, $\mathbb{S}^{+0-}$, and $\mathbb{S}^{0--}$, called *future, present,* and *past.* In addition there are now two invariant disjoint, still lower dimensional cones $\mathbb{S}^{+00}$ and $\mathbb{S}^{00-}$, the *future* and *past doubly null cones.* Finally, there is the one-point triply null cone, the origin $\mathbb{S}^{000}$, for a total of ten invariant cones in the ternary time space. (In the N-ary case there are $(N+2)!/[3!(N-1)!]$ invariant light cones of all dimensions.)

20. Causality. Important: The causal structure of the binary tangent space is automatically preserved in the transition from binary to ternary, in that there is only one timelike dimension among the N^2, and past and future are connected neither topologically nor by a symmetry transformation in A_2. Indeed, the timelike future cone $\mathbb{S}^{+++}$ is connected, and its boundary is the union of the eight-dimensional future null cone $\mathbb{S}^{++0}$, the future doubly null cone $\mathbb{S}^{+00}$, and the origin $\mathbb{S}^{000}$. Any curve joining a timelike future $\mathbf{t}$ to $-\mathbf{t}$ must cut this boundary. As for symmetry, each of the four nine-dimensional cones is invariant under A_2.

Riemannian hyperspaces need special assumptions to maintain causality, just as a special assumption was necessary to provide it in the first place. Spin manifolds are naturally causal in every dimension.

21. The chronometric form. The fundamental symmetric chronometric form $\delta_{\alpha\beta\gamma}$ of the nine-dimensional ternary space $\mathbb{S}_3$ is not quadratic like that of the binary spinor time space, but cubic:

$$\|\mathbf{t}\|_3 = 3!\,\det(t) = \delta_{\alpha\beta\gamma}t^\alpha t^\beta t^\gamma,$$

$$\delta_{abc} := \varepsilon_{AB\Gamma}\varepsilon_{\dot{A}\dot{B}\dot{\Gamma}}/3!$$

The coordinatization

$$\mathbf{t} = \begin{bmatrix} t+z & x-iy & a-ib \\ x+iy & t-z & c-id \\ a+ib & c+id & e \end{bmatrix}$$

of the ternary Hermitian matrices by the nine coordinates t, x, y, z, a, b, c, d, e is best adapted to the ordinary t, x, y, z time space. This $e := t^i$ is a null coordinate. The binary identity $t = \mathrm{tr}_2(\mathbf{t})/2$, however, suggests that another ternary coordinatization with $t = \mathrm{tr}_3(\mathbf{t})/3$, such as

$$\mathbf{t} = \begin{bmatrix} t+z+e & x-iy & a-ib \\ x+iy & t-z+e & c-id \\ a+ib & c+id & t-2e \end{bmatrix}$$

will be more useful when the external time space does not break the symmetry. This e is not a null coordinate.

Proper time. The ternary *proper time* τ of a future timelike $\mathbf{t}$ is defined by

$$\tau = \|\mathbf{t}\|^{1/3}$$

A *future timelike curve* is one whose tangent $d\mathbf{t}$ lies in $\mathbb{S}^{+++}$. The proper time of such a curve is given by

$$\tau = \int d\tau = \int \|d\mathbf{t}\|^{1/3}$$

The ternary determinantal form det(**t**) vanishes for ordinary (binary) time space directions.

Orthogonality and duality. In binary geometry, orthogonality is a dyadic relation $\mathbf{s}\perp\mathbf{t}$ or $\perp(\mathbf{s}, \mathbf{t})$; in the ternary geometry, it is a triadic one, defined by

$$\perp(\mathbf{s}, \mathbf{t}, \mathbf{u}) := \delta_{\alpha\beta\gamma} s^\alpha t^\beta u^\gamma = 0$$

In binary geometry, contracting a raised index with the fundamental form generates one lower index; in the ternary, two. For any vector $\mathbf{t} = (t^\alpha)$ we define a dual vector $\mathbf{t}^{\mathrm{D}} = (t_\alpha)$, generalizing the binary concept, with

$$t_\alpha := \delta_{\beta\gamma} t^\beta t^\gamma$$

Then

$$\mathbf{t}^{\mathrm{D}} \cdot \mathbf{t} := t_\alpha t^\alpha = \|\mathbf{t}\|_3$$

The dualization process $\mathbf{t}^{\mathrm{D}}$ for vectors induces one for arbitrary tensors in a unique natural way, maintaining this relation between dual and norm. For example, $\boldsymbol{\partial}^{\mathrm{D}}$ is a second-order differential operator. The dual form $\boldsymbol{\delta}^{\mathrm{D}} = (\delta^{abc})$ has the property

$$2\delta^{\alpha\beta\gamma} u_\alpha u_\beta u_\gamma = \det(u)$$

where $u_\alpha = u_{\dot{A}A}$ is any covector (dual vector). The ternary norm $\| \ldots \|_3$, like the binary one, is extended to arbitrary tensors $T\ldots$; but now it is a contraction of a product of *three* $T\ldots$'s with as many $\delta_{\alpha\beta\gamma}$ and $\delta^{\alpha\beta\gamma}$ as required for a scalar.

In variational calculations it is useful to write the variation of $\|\mathbf{t}\|_3$ due to a variation $\delta\mathbf{t}$ of $\mathbf{t}$ in the form

$$\delta\|\mathbf{t}\|_3 = 3\mathbf{t}^{\mathrm{D}} \cdot \delta\mathbf{t}$$

The vector formed from any two external covectors by contracting them with the ternary fundamental form must be internal, and parallel to the internal null vector; it is easiest to see this in a normal frame. The four-dimensional subspace $\mathbb{S}_2 \subset \mathbb{S}_3$ thus defines a unique internal direction in the nine-dimensional ternary imbedding space $\mathbb{S}_3$.

This cannot happen in a quadratic geometry of the same dimension, for there the subgroup that fixes a four-dimensional linear subspace fixes no direction. Such deviations of ternary geometry from binary intuition must be expected because the ternary group $SL(3, \mathbb{C})$ has so many fewer parameters than the group $SO(9, \mathbb{R})$ of a quadratic form (16 versus 36).

22. Stage iii. The manifold. The world is now postulated to be a ternary spin manifold $\mathfrak{S}_3$. It therefore possesses at each point p:

1. A local spinor space $\Sigma^3(p)$.
2. A local tangent space $d\mathfrak{S}_3(p)$ isomorphic to $\mathbb{S}_3$ and an isomorphism σ, the *spin map*, inducing:
3. A real symmetric **cubic** form $g_{abc}(p)\,dt^a dt^b dt^c$, with coefficients $g_{abc} = \delta_{abc}$ in a normal coordinate system, inducing a local norm $\|\ldots\|_p$ for polyspinors. Since the potentials can be transformed to the constant determinantal form δ by an element of the 81-parameter group $GL(9, \mathbb{R})$, and the transformation of vectors leaving this form invariant include the 16-parameter subgroup A_2, the number of *independent* potentials among the 165 is at most $81 - 16 = 65$. There are at least 100 relations.
4. A covariant derivative $\mathbf{D} = (D_a)$ generalizing the differential operator $\boldsymbol{\partial}$; a **3**-spinor *connection* $\Gamma^A{}_{Bc}(p)$ reducing to 0 at p in a normal coordinate system at p; and spinor curvature $R^A{}_{Bcd}(p)$. The spinor connection and curvature induce the vector connection $\Gamma^a{}_{bc}$, curvature $R^a{}_{bcd}$, and Ricci tensor $R_{ab} := R^c{}_{acb}$ in a routine way.

The manifold $\mathfrak{S}_3$ need not be homeomorphic to a linear space nor a topological product of internal and external spaces (like $\mathbb{S}_3$), and if it is such a product, the internal space need not be topologically trivial.

Since ten g's are ordinary external gravitational potentials, $65 - 10 = 55$ are new. They have binary spins 0, 1, 2, and 3, since they have 0, 1, 2, or 3 external indices.

The nature of the potentials is the main difference in form between the spin theory and the Riemannian one: The Riemannian theory adheres to the quadratic chronometric form, adding $45 - 10 = 35$ new potentials to the ordinary gravitational potentials in going from four to nine time space dimensions, with spins 0 and 1 only. Indeed, for a given dimension the *vector* connection $\Gamma^a{}_{bc}$ and curvature $R^a{}_{bc\delta}$ have the same symbol and transformation law in a spin manifold as in a Riemannian one.

23. Reduction from cubic to quadratic chronometric. The nine-dimensional world splits into external and internal differently in classical and quantum theories. In classical theories, it is not inappropriate to think of the binary world as a submanifold of the ternary one, as if its particles were localized at $a = b = c = d = e = 0$. In quantum theories, such localization would violate uncertainty relations, but a constraint may limit the internal momentum components if these are conserved: $p_a = \cdots = p_e = 0$. Then the binary external world is that of quanta totally unlocalized in internal space.

As 3-spinors approach 2-spinors, $\psi^3 \to 0$, the corresponding ternary vectors $\mathbf{t}$ approach binary vectors $\mathbf{t}_0$:

$$\mathbf{t} = \mathbf{t}_0 + e$$

where $\mathbf{t}_0$ is an external vector and $\mathbf{e}$ is an internal vector, with $\mathbf{e} \to 0$. The determinantal relation

$$\begin{vmatrix} t+z & x-iy & 0 \\ x+iy & t-z & 0 \\ 0 & 0 & 1 \end{vmatrix} = t^2 - x^2 - y^2 - z^2 = \begin{vmatrix} t+z & x-iy \\ x+iy & t-z \end{vmatrix}$$

implies that

$$\|\mathbf{t}_0\|_2 = \|\mathbf{t}_0 + \mathbf{o}\|_3$$

where $\mathbf{o}$ is the internal null vector. Expanding this ternary determinant shows that for any external vector $dt_0{}^a$,

$$g_{ab} dt_0{}^a dt_0{}^b = 3 g_{abc} dt_0{}^a dt_0{}^b o^c$$

Thus, the usual binary quadratic chronometric is a component of the ternary cubic chronometric along an internal null direction.

24. Dual potentials. The binary algebraic relation between the gravitational potentials g_{ab} and the dual potentials g^{ab} is

$$g_{ab} g^{bc} = \delta_a{}^c$$

One ternary correspondent to this binary relation is

$$g_{abc} g^{bcd} = 4 \delta_a{}^d$$

Therefore when we raise a ternary vector index to a pair and then lower the pair, we get back the original tensor with a factor of 1/4.

In any normal coordinates in a ternary manifold (where $g_{abc} = \delta_{abc}$) the relation $g^{abc} = \delta^{abc}$ also holds. This linkage between ternary form and co-form corresponds exactly to the binary fact that in any frame where the binary form $g_{ab} = \delta_{ab}$, the dual form $g^{ab} = \delta^{ab}$; this serves to define the dual potentials in terms of the potentials as uniquely as the algebraic relations do.

Independent variables. In Riemannian geometry it is assumed that the binary fundamental form is covariantly constant with respect to the vector connection: $D_a g_{bc} = 0$. The 40 derivatives of the ten gravitational potentials then determine the 40 components of the symmetric part of the vector connection, according to the Christoffel relations, and can themselves still take on arbitrary initial values. In the ternary case, however, there are 9×45 connection components, but some 9×65 derivatives of the 65 potentials. Covariant constancy of the ternary cubic fundamental form would not only determine the relevant connection components, but would also impose up to 9×20 nontrivial relations among the derivatives of the ternary potentials; these derivatives could not then take on arbitrary initial values. It seems unlikely, therefore, that the ternary potentials are covariantly constant in general. The ternary gauge field $\Gamma^a{}_{bc}$ and potential field g_{abc} are probably independently variable, as Einstein prefers in the binary case ("Palatini device").

25. Stage iv. Dynamics. The Hilbert action R does not generalize simply from binary to ternary spin, though the binary R may appear as one term among others in the reduction of a ternary action to binary time space. The nearest ternary kin to the Hilbert action with cosmological term is

$$A = (R_{ab} R_{cd} R_{ef} g^{abc} g^{def} + \Lambda) g^{1/3}$$

The ternary generalization A_3 of the binary quadratic action A_2 is the ternary *cubic action*

$$A_3 := \| R^{abcd} \|_3$$

The ternary determinantal action is even closer in form to its binary ancestor:

$$A_\Gamma := \det(R_{ab})$$

When we do not constrain the potentials to be covariantly constant, the total action density may also include a term

$$A_g := \| D_a g_{bcd} \|_3$$

trilinear in the potential gradients. A_g is analogous to the $(D\eta)^2$ term of quaternion quantum field theory and the Higgs term of electroweak unification theory. Like them, A_g reduces the symmetry of the theory and imparts mass to some otherwise massless modes of the gauge field. In the present context, this symmetry-breaking is relevant to the problem of dimensional reduction. The consequences of these actions are under study.

While the invariant actions we are considering are classical, we take a nonclassical view of them. The classical view, represented by Einstein and Eddington, is that there exist fundamental fields obeying kinematical and dynamical laws so beautiful that they could credibly be the Law of Nature, the blueprint followed in the creation of the universe. Presumably no fundamental fields exist; and the kinematical and dynamical descriptions usually called laws are phenomenological, not fundamental, and current inventories of an ongoing process, not the original blueprints for a primeval one. This implies that symmetries of these actions are not exact and absolute, but conditional.

26. Ternary wave equations. We begin with fields in the flat ternary space $\mathbb{S}_3$. These have no immediate physical application, since the present world has such strong internal curvature, but all the differential operators that appear have meaning in the tangent space to a $\mathfrak{S}_3$.

Trine-Gordon equation. The unique scalar wave equation of least differential order, analogous to the binary d'Alembert equation, is

$$\det(\boldsymbol{\partial})\varphi(\mathbf{t}) = 0$$

for a real or complex field φ on dT_3. This has plane wave solutions

$$\varphi(\mathbf{t}) = \varphi(0)\exp(-i\boldsymbol{\omega}\cdot\mathbf{t})$$

where $\boldsymbol{\omega} = (\omega_{\dot{B}B})$ is a constant wavevector belonging to the dual space $\mathbb{S}_3{}^{\mathrm{D}}$ and obeying the ternary dispersion relation

$$\det(\boldsymbol{\omega}) = 0$$

This theory reproduces the binary dispersion relation, and hence, effectively, the d'Alembert wave equation of ordinary special relativity, when the internal component of $\boldsymbol{\omega}$ is parallel to the internal null covector and $\omega_{\dot{3}3} \neq 0$.

The next simplest equation, the *trine-Gordon* equation, has a mass term:

$$[\det(\boldsymbol{\partial}) - iM^3]\varphi(\mathbf{t}) = 0$$

The dispersion relation for its plane waves is

$$\det(\boldsymbol{\omega}) = M^3$$

This is the relation between mass and energy of ternary relativity.

Trino. The trino, the ternary Weyl neutrino, obeys

$$\boldsymbol{\partial}\nu(\mathbf{t}) = 0$$

where $\nu(\mathbf{t}) = (\nu^A(\mathbf{t}))$ is a ternary spinor amplitude depending on the time space variable $\mathbf{t}$. Since $\det(\boldsymbol{\partial}) = \boldsymbol{\partial}^{\mathrm{D}}\cdot\boldsymbol{\partial}$, this spinor equation implies that the trine-Gordon wave equation holds for each component of ν.

Trirac equation. To introduce a mass term into the massless equation, we equate $\partial_{\dot{A}B}\nu^B$ to a multiple of another spinor μ, which must therefore have the index structure $\mu_{\dot{A}}$:

$$\boldsymbol{\partial}\nu = M\mu$$

To close the system, we apply a differential operator to μ and equate the result to a multiple of ν. The differential operator must therefore have the index structure $d^{\mathrm{A}\dot{\mathrm{B}}}$. The invariant differential operator of lowest order with this structure is $\boldsymbol{\partial}^{\mathrm{D}}$. The trirac equation is therefore

$$\boldsymbol{\partial}\nu = M\mu, \qquad \boldsymbol{\partial}^{\mathrm{D}}\mu = iM^2\nu$$

The mass squared, M^2, is necessary on dimensional grounds; in the ternary theory $\boldsymbol{\partial}^{\mathrm{D}}$ is a second-order differential operator. Only in the binary theory do these two equations combine into one of the first-order Dirac form $(\boldsymbol{\gamma} - \boldsymbol{\partial} + M)\psi = 0$. In the N-ary theory, the equation for two spinors mixes differential orders 1 and $N-1$.

The coefficients of this massive spinor wave equation have been chosen so that the components of μ and ν obey the trine-Gordon wave equation for mass M.

A trirac equation closer in spirit to the Dirac would have the form

$$(\gamma^a\partial_a - M)\psi = 0$$

with coefficients γ^a obeying

$$(\gamma^a\partial_a)^3 = \det(\partial)$$

In the binary case it is well known that the corresponding condition

$$(\gamma^a\partial_a)^2 = \det(\partial)$$

with a 2×2 determinant leads to a finite-dimensional algebra of γ matrices, the Dirac Clifford algebra. We do not expect this for $N > 2$.

Minimal coupling. To generalize these equations from the flat space $\mathbb{S}_3$ to a ternary manifold $\mathfrak{S}_3$, and at the same time provide a minimal interaction with the gauge field, we replace the ordinary by the covariant spinor derivative:

$$\boldsymbol{\partial} \to \mathbf{D}$$

27. Symmetry group and multiplet structure. The subgroup of $SL(3, \mathbb{C})$ of all elements that fix every external spinor, and therefore also fix every point of dT_2, is the four-parameter *internal* group G_i of matrices g of the form

$$\begin{bmatrix} 1 & 0 & \alpha \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{bmatrix}$$

Setting $\psi = \alpha\uparrow + \beta\downarrow$, we may write this matrix as

$$g = \mathbf{1} + \psi(\rightarrow) + (\rightarrow\rightarrow^{D}) =: g(\psi)$$

Since $g(\psi)g(\psi') = g(\psi + \psi')$, the multiplicative group G_i is product of the additive group Σ^2 of the 2-spinors ψ, isomorphic to the Abelian group $\mathbb{R}^4$.

Any binary tangent vector can be diagonalized by a binary transformation. It is easy to show that any ternary tangent vector may be diagonalized by a suitable product of binary transformations acting on the external space only and internal transformations. The norm of a diagonal tangent vector $\operatorname{diag}(t_+, t_-, t_i)$ is

$$\|\mathbf{t}\| = 6t_+t_-t_i$$

Similarly for wave vectors $\boldsymbol{\omega}$; diagonalization puts the external direction of propagation along the z axis and the internal direction of propagation along the internal null direction. The norm

$$\|\boldsymbol{\omega}\| = 6\omega_+\omega_-\omega_i = 6(\omega^2 - k^2)k_i$$

is the Minkowski norm of the external wave vector multiplied by the internal wave number.

In Riemannian theories, the time and space wave numbers of the modes are thought to combine as a quadratic sum with appropriate signs to produce the square of the total rest mass. In N-ary physics, the modes are neither timelike nor spacelike, but null, and combine multiplicatively. The quadratic addition recipe appears as an artifact of four dimensions, where $\omega_+\omega_- = \omega_0{}^2 - \omega_3{}^2$.

The descending mass spectrum. In consequence, we do not get the correspondence limit (in this flat case, at any rate) by setting the internal wave number equal to zero. That is not even a possible mode unless the parameter M is zero, too, and then it produces a continuous spectrum of all masses. Instead, we must single out a definite internal periodicity k_i or a discrete spectrum of k_i. Then the mass spectrum in flat, ternary time space is

$$(\omega^2 - k^2) = M^3/k_i$$

This multiplicative factor in the mass spectrum is pleasantly suggestive of the hierarchy of particles, but we also see here what might be the first sign of a fatal defect inherent in the theory:

If the spectrum of internal wave numbers is unbounded, there is *a descending mass spectrum with* 0 *as a point of accumulation.*

The theory will survive this apparent discord with experience only if high internal curvature changes the result or its experimental meaning.

Unitary symmetry. The idea behind hypervector, hyperspin, and paraspin theories alike is to account for an empirical group structure of particle interactions by introducing an internal substratum of which the empirical group G is a symmetry. Ternary geometry, however, does not of itself nominate the group A_2 of ternary spinors as a candidate for a particle symmetry group, since A_2 is strongly broken by the difference in curvature between the internal and external spaces. Color SU_3, for example, relates three external spinors. The group A_2 relates two external spinors $\uparrow$ and $\downarrow$ to the internal spinor $\rightarrow$, which is a scalar under A_1.

Since the three creation operators for the three modes $\uparrow$, $\downarrow$, and $\rightarrow$ presumably anticommute, this triplet apparently violates the connection between spin and statistics. The first two modes, however, propagate in external directions, and the third propagates in internal directions and has ultrahigh energy. Therefore, no observable violation of the spin-statistics connection should occur.

Only at ultrahigh energies, which might wash out the initial time space curvature, could A_2 manifest itself as a symmetry of ternary dynamics; then the concept of binary spin disappears with the distinction between internal and external, and ternary spin takes its place. Otherwise, A_2 appears only as a group that can act on the fundamental variables, not as a symmetry of the Hamiltonian. Again no observable violation of spin-statistics is to be expected.

The idea of Kaluza-Klein-De Witt theories, transcribed to ternary geometry, is that the world may not be an $\mathfrak{S}^4$ with gauge fields, but may be a product

$$\mathfrak{S}^9 = \mathfrak{S}^4 \otimes I^5$$

of an external $\mathfrak{S}^4$ of comparatively small curvature and a compact five-dimensional internal space I^5 of high curvature, with an action of the symmetry group G on $\mathfrak{S}^9$ that fixes each point of $\mathfrak{S}^4$ and maps I^5 into itself. I^5 may, for example, be homeomorphic to the group manifold G itself, or a coset space thereof, or a product of these group spaces with a "radial" space.

One promising candidate for an internal space supporting SU_3 is $I^5 = S^5$, the five-dimensional sphere of unit vectors in H^3, the three-dimensional complex Hilbert space, with G acting on I^5 as unitary 3×3 complex matrices act on the unit sphere $S^5 \subset H^3$. This is not the natural action of A_1 on this S^5; there are other five-dimensional compact spaces that I^5 might turn out to be *a priori*; and there are other invariance groups that might belong to I^5 even when I^5 turns out to be S^5, such as SO_5. If there is any natural theory of SU_3 symmetry within the ternary geometry, It therefore rests on solving the dynamical equations for the chronometric form to obtain a particular solution with a suitable topology (such as $dT_2 \otimes S^5$) and with SU_3 symmetry. It seems to have little or nothing to do with the mere presence of SU_3 in A_2.

While the group A_{N-1} of the N-ary tangent space $\mathbb{S}^n$ is not an internal symmetry group, the internal group G_i of $\mathbb{S}^n$ might be one. It may be possible to construct standard manifolds having the internal group as an internal symmetry group; for example, the product of a flat, external space invariant under G_i and an internal space that is homogeneous under G_i. It is therefore useful to examine manifolds whose internal group includes SU_3. The internal group of the ternary geometry, we have seen in this paragraph, is Abelian and six-dimensional; it does not include SU_3. The first N-ary geometry whose internal group includes SU_3 is evidently the quintary, founded on 5-spinors and the group $SL(5, \mathbb{C})$; its internal group includes $SL(3, \mathbb{C})$ as well as SU_3.

[Quantum set theory already suggests that the kets of the world be assembled out of products of five Grassmann elements representing elementary quantum simplices of four dimensions. The group of such a pentad is $GL(5, \mathbb{C})$, and therefore the continuum approximation to such a simplicial complex might have this as its local group.]

4. DISCUSSION

28. Main results. The spinor imbedding strategy generates a less arbitrary revision of time space geometry than the Riemannian. The N-ary hyperspace, the imbedding time space, can have only the quadratic dimensions $n = 1, 4, 9, 16, \ldots$ resulting from spinors of N components with $N = 1, 2, 3, 4, \ldots$. This excludes Kaluza's one-dimensional internal space, among with many others, since with four external coordinates the minimum number of internal coordinates is $9-4=5$. The N^2-4 internal space dimensions all prove to be necessarily spacelike, maintaining the strict causal split between past and future. Most radical revision of all, the chronological structure of the hyperspin manifold of dimension $n = N^2$ is not given by a symmetric n-ary quadratic form, as has almost always been assumed. The degree N of the chronometric form uniquely determines the dimensionality n of the total time space by the quadratic relation $n = N^2$. The chronometric structure is defined by a symmetric n-ary N-ic form (what Sylvester calls a *quantic*)

$$d\tau^N = g_{ab\ldots z} dx^a \cdots dx^z$$

with N n-valued indices $A, \ldots, Z = 1, \ldots, n$, which is a quadratic form only for $N = 2$ (not the most general n-ary N-ic quantic, however, but one having determinantal normal form).

In hyperspin geometry it is no longer possible to make a scalar from two spinors. In the simplest such theory, the ternary, we need at least three spinors, or a spinor and a dual spinor, to make a scalar; just as we need three colored quarks or a quark and antiquark to make a colorless quantum. Nor can a scalar be made from two vectors; it takes three vectors, or a vector and a covector, to make a scalar. The bilinear inner product of two ordinary vectors is not a scalar, but a multiple of a unique null codirection, the *internal* null covector of the ternary space.

Wave equations and parity violation. The ternary Pauli matrices are 3×3 and there are nine of them. The dispersion relation of the trino is $\det(\boldsymbol{\omega}) = 0$.

The ternary scalar wave equation, the trine-Gordon equation, is of the third differential order instead of the second.

The massive spinor equation, the trirac equation, sprouts a surprising asymmetry. Instead of being uniformly of the first differential order like the Dirac equation, the trirac equation consists of a first-order equation for (say) a left-handed spinor and a second-order equation for a right-handed spinor. In the N-ary geometry the orders are 1 and $N-1$. It is only in the binary case $N=2$ that $N-1=1$ and the massive spinor equation treats left- and right-handed spinors symmetrically. In general one of the binary spinor symmetries that interchanges right- and left-handed spinors disappears.

Nevertheless, according to the hyperspin correspondence principle, we expect the trirac equation to go over into the Dirac equation, with perfect left–right symmetry, as the internal time scale $\varepsilon \to 0$. Therefore, for small, nonzero ε there should be small parity violations. These and other applications of N-ary manifolds will be taken up in a subsequent paper.

It is impossible not to wonder at least briefly whether the case $N=3$ might have special connection to quark color or flavor, since $A_2 \supset SU_3$. We have considered and discounted this connection in paragraph 27. A suitable ternary spin manifold $\mathfrak{S}_3$ may still have an internal SU_3 symmetry not connected to the group $SL(3, \mathbb{C})$ of the geometry, but we suggest $\mathfrak{S}_5$ for a more promising hyperspin theory of SU_3.

29. Implications. A nonquadratic chronometric has been considered before, for example, in Finsler spaces. This previously seemed to be a distortion of the Riemann theory that was motivated mainly by a desire for the utmost generality. In the N-ary spin manifolds, however, we have a quite special line of higher dimensional geometries that are simpler and more specialized to the needs of physics than the older line of Riemannian geometries, and beautiful in their economy of means. The possibility that hyperspace has an N-ic rather than a quadratic chronometric form must be considered.

30. Limitations. Nevertheless, it is probable that hyperspin geometries, too, are smoothed classical continuum approximations to some still unknown quantum theory of discrete elementary processes, and will be unable to describe cosmogony and the deep interior of physical particles except as singularities. Further, if the internal perimeter of the world is of the order of the Planck length, a classical theory of time space such as the present one has value only as a general indication of what might happen, and we must return to the search for a quantum theory of time space.

31. Generalizations. There are as many generalizations of the present theory as there are groups of linear transformations containing $A_1 = C_1 = D_1$. In particular, since we have now used Cartan's A's, B's and D's, one might explore a line of spinor-based manifolds using the C's, giving hyperspinors an invariant symplectic form.

The four-dimensional Minkowskian manifold of classical gravity theory seems now like a kind of Time's Square, where three main lines meet. Of these the B/D line is the oldest and best explored one. In this paper we discover (we think) the A train and the C train, and take the A train to the next station, the ternary spin manifold.

We choose the A line for our first exploration not so much for its elegance as for its connection to quantum set theory, which describes the world (or the system under study) by kets in a Grassmann algebra (with additional hierarchic structure). The simplest representation of the Lorentz group in this algebra is A_1 acting as 2×2 linear transformations of a pair of Grassmann elements; but there is no permanent wall separating two elements from the rest of the infinite-dimensional Grassmann algebra of quantum set theory, and as other elements join them they carry us naturally along the sequence of groups $A_2, A_3, \ldots$ of the A line. This is the origin of the present work. There is no equally simple connection to either the B/D line or the C line of manifolds.

Nevertheless, the C train, too, is worth an inspection. Its spinor symplectic form endows its time space vectors with a symmetric quadratic form like that on the B/D line, and it has in addition an N-ic form like that on the A line. Spin manifolds on this line therefore admit a first-order Dirac equation. The signature of the quadratic form is not of the causal kind, but perhaps internal time loops are not disastrous as long as the external space is causal.

32. The missing quantum foundations. The classic binary constructions of stage ii of paragraphs 13 and 14 that we have generalized here have themselves long cried out for an explanation—in vain.

(The matrix t of stage ii resembles a statistical matrix strongly. While the fundamental spinor components may be quantum amplitudes, the matrix t is a classical coordinate, measured in a single experiment. Is a timelike future vector a statistical description of an assembly of spinorial quantum entities constituting the chronometric manifold? What is the underlying quantum theory? What does the determinantal norm of the time space vector in stage ii say about the underlying quantum entities?)

In this paper we have briefly turned away from the levels beneath time space and toward those above.

33. Duality and triality. Dualities pervade theoretical physics. Some physical dualities have already been traced to the quantum twoness of our spinors (the covalent bond, for example). We see in this work that metrical dualities like the bilinearity of our free Lagrangians, the dyadic relation of orthogonality, and the duality relation between vectors, could also stem from this root. If ternary hyperspin exists, as we propose, then each of these dualities is but a part of a deeper triality, one of whose elements lies in internal space and has been unnoticed until now.

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